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Winding strings and AdS_3 black holes

Jan Troost
Center for Theoretical Physics, MIT
77 Mass Ave, Cambridge, MA 02139, USA
E-mail: troost@mit.edu

Abstract: We start a systematic study of string theory in AdS_3 black hole backgrounds. Firstly, we analyse in detail the geodesic structure of the BTZ black hole, including spacelike geodesics. Secondly, we study the spectrum for massive and massless scalar fields, paying particular attention to the connection between $\mathrm{Sl}(2,\mathbb{R})$ subgroups, the theory of special functions and global properties of the BTZ black holes. We construct classical strings that wind the black holes. Finally, we apply the general formalism to the vacuum black hole background, and formulate the boundary spacetime Virasoro algebra in terms of worldsheet operators. We moreover establish the link between a proposal for a ghost free spectrum for $\mathrm{Sl}(2,\mathbb{R})$ string propagation and the massless black hole background, thereby claryfing aspects of the AdS_3/CFT correspondence.

Keywords: Conformal Field Models in String Theory, Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Black Holes.

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Contents

1. Introduction

In the past ten years, we have learned a lot about string theory in black hole backgrounds. In [1] a two-dimensional black hole background was analysed, using an exact coset conformal field theory. It was shown that the causal structure of the spacetime mimicked well its four-dimensional analogue. Using the powerful tools of two-dimensional conformal field theory, the physics (scattering, particle production) of the black hole background was further analysed in [2].

We believe it is important to further develop the analysis of string theory on black hole backgrounds, starting from first principles. Indeed, since string theory is a consistent, unitary theory of quantum gravity, it should be able to shed light, rigorously, on fundamental problems like black hole entropy and the black hole information paradox. Significant progress was made on both fronts (see e.g. [3], and [4]), mostly using our improved understanding of non-perturbative D-brane physics and supersymmetry. But we still lack a universal insight into the microstates that make up the black hole entropy, and a clear resolution of the information paradox.

Thus it remains important to analyse simple black hole backgrounds in string theory, and to uncover more of the quantum gravitational stringy aspects of singular backgrounds. The AdS_3 (or BTZ) black hole [5, 6] presents itself as a good candidate for further study. Firstly, it arises in string theory from a near-horizon limit of (for example) an $F1-NS5-$ momentum system. The three-dimensional black hole is therefore a factor of a spacetime with consistent string propagation. Moreover, the BTZ black hole is locally AdS_3. String propagation on AdS_3 backgrounds has been analysed in great detail, in the wake of the AdS/CFT correspondence (see e.g. [7, 8, 9]). Even more importantly, the detailed and rigorous analysis of $\mathrm{Sl}(2,\mathbb{R})$ and $\mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2)$ conformal field theory in [10] and [11, 12, 13] has provided us with an entirely consistent picture of string propagation on AdS_3 backgrounds, and some more insight into the dual conformal field theory. Now, since the BTZ black hole differs from AdS_3 by global identifications only [6], we can view string propagation on BTZ black holes as a discrete orbifold of string propagation on AdS_3 [14, 15]. Thus, in principle, it seems that one can rigorously address fundamental questions on black hole physics in string theory using these backgrounds. More ground needs to be covered though, before a detailed analysis of the lorentzian $\mathrm{Sl}(2,\mathbb{R})$ or euclidean $\mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2)$ orbifold CFT is feasible. We take a few first, systematic, steps towards this goal.

The plan of our paper is the following. In section 2, we review the backgrounds that we are interested in, i.e. the space of BTZ black holes. Then we study the dynamics of classical particles in BTZ backgrounds in section 3, and make a connection to the classical dynamics of strings. Next we concentrate on scalar fields and show how the analysis on a generic BTZ black hole simplifies when an extremal limit is taken. This nicely fits into special function theory and will allow us to connect to $\mathrm{Sl}(2,\mathbb{R})$ representation theory later. We then go on to link the global properties of BTZ black hole backgrounds with preferred parametrisations of the $\mathrm{Sl}(2,\mathbb{R})$ group manifold (and comment briefly on a recent cosmological application). In section 7, we concentrate on the massless black hole background and analyse a few aspects of the string worldsheet CFT in this background. In particular, we are able to

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construct the boundary spacetime conformal algebra in terms of operators on the string worldsheet (in a free field approximation), and we show the connection between winding strings in the vacuum black hole background and a proposal for a ghost free spectrum for strings on $Sl(2,\mathbb{R})$. We conclude and point out more open problems in section 8.

2. BTZ black holes

The low-energy description of closed superstrings is supergravity. The supergravity equations of motion have solutions that include a metric factor that is the BTZ black hole. These solutions can be obtained, for instance, by taking the near-horizon limit of (macroscopic) fundamental strings, wrapping an S^1 of a T^5, NS5-branes that wrap the torus, and momentum modes along the fundamental strings [16]–[19] (or a U-dual thereof). Our main interest is in the non-trivial physics associated to the factor in the metric that corresponds to a BTZ black hole [5]. We therefore briefly review the space of BTZ black hole metrics.

The BTZ black holes in three-dimensional anti-de Sitter space are parametrized by their mass M and angular momentum J. The metric and global identification are given by:

$$\begin{aligned} ds^2 &= -\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)dt^2+\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)^{-1}dr^2+ \\ &\quad + r^2\left(d\phi-\frac{J}{2r^2}dt\right)^2 \end{aligned} \tag{2.1}$$

where $\phi\in[0,2\pi[$. The mass M and angular momentum J of the black hole are expressed in terms of the outer and inner horizon as $M=\frac{r_+^2+r_-^2}{l^2}$ and $J=\pm\frac{2r_+r_-}{l}$. The inner and outer horizon are then located at the positive square root of:

$$r_\pm^2=\frac{Ml^2}{2}\left(1\pm\left(1-\left(\frac{J}{Ml}\right)^2\right)^{\frac{1}{2}}\right). \tag{2.2}$$

The all-important global properties of the BTZ black holes were analysed in detail in [6].[1]

Supersymmetry. In string theory, we are often interested in backgrounds that preserve some supersymmetry, for stability and simplicity. The supersymmetric BTZ black holes [20] are the extremal ones. These are the BTZ vacuum black hole with $M=0=J$, and the extremal black hole with mass $M=|J|$ different from zero. Later, we will focus on the vacuum BH. We note that anti-de Sitter space is formally obtained as the BTZ black hole with $M=-1$ and $J=0$. These facts yield figure 1, depicting the space of BTZ black holes.

3. Geodesics

As a preliminary to studying classical strings in the BTZ black hole background, we study the geodesic structure of the black hole. The geodesics have been studied in detail before, in [21, 22]. Our motivation to revisit the problem is twofold. First of all, we want to

[1] We note moreover that we will always suppose that the background NSNS twoform B is such as to complete the 2d WZW conformal field theory ($B=1/2r^2d\phi\wedge dt$) on the string worldsheet [14].

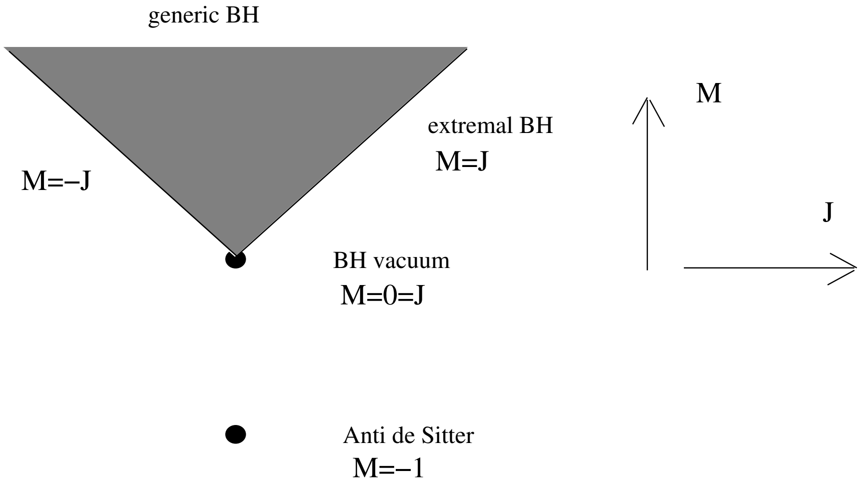

Figure 1: The space of BTZ black holes

extend the study in the literature to include spacelike geodesics. On the one hand, that makes sense because of the fact that negative mass excitations that do not violate the Breitenlohner-Freedman bound [23, 24] do not give rise to instabilities in asymptotically AdS spactimes. On the other hand, spacelike geodesics proved to be a good starting point for finding long, winding strings that turn out to have positive mass. A second reason for revisiting the nice analysis in [22] is the particular rescaling of variables that is used, which is not appropriate for describing the geodesics of the supersymmetric vacuum black hole.

We briefly review timelike and lightlike geodesics [22] and study spacelike geodesics in the BTZ background. We define the conserved energy and angular momentum (associated to the two Killing vectors (∂_t and ∂_ϕ) of the metric (2.1)) as [22]:

$$\begin{aligned} E &= \left(-M + \frac{r^2}{l^2}\right)\frac{dt}{d\tau} + \frac{J}{2}\frac{d\phi}{d\tau} \\ L &= r^2\frac{d\phi}{d\tau} - \frac{J}{2}\frac{dt}{d\tau}\,. \end{aligned} \tag{3.1}$$

In terms of these conserved quantities, we can write the geodesic equations as:

$$\begin{aligned} \dot{t} &= l^2\frac{Er^2 - \frac{JL}{2}}{(r^2 - r_+^2)(r^2 - r_-^2)} \\ \dot{\phi} &= l^2\frac{(\frac{r^2}{l^2} - M)L + \frac{JE}{2}}{(r^2 - r_+^2)(r^2 - r_-^2)} \\ r^2\dot{r}^2 &= -m^2\left(\frac{r^4}{l^2} - Mr^2 + \frac{J^2}{4}\right) \\ &\quad + \left(E^2 - \frac{L^2}{l^2}\right)r^2 + L^2M - JEL \end{aligned}$$

where τ is the eigentime, and m is the mass of the particle. For convenience, we furthermore define the quantities:

$$\begin{aligned} \alpha &= E^2 - \frac{L^2}{l^2} - Mm^2 \\ \beta &= L^2 M - \frac{1}{4} m^2 J^2 - JEL\,. \end{aligned} \tag{3.2}$$

We put $l = 1$ from now on, and define a coordinate $y = r^2$ in terms of which the radial equation simplifies. Then we can rewrite the radial differential equation as:

$$\dot{y}^2 = 4m^2 \left(-y^2 + \frac{\alpha}{m^2} y + \frac{\beta}{m^2}\right). \tag{3.3}$$

We have $m^2 > 0$ for timelike geodesics, $m^2 = 0$ for lightlike geodesics and $m^2 < 0$ for tachyons. In the following we solve the radial equation (3.3) for the three different cases. (For further straightforward integration of the angular and time equation see [22].)

3.1 Massless particles

For lightlike geodesics the equations simplify:

$$\begin{aligned} \dot{y}^2 &= 4\alpha y + 4\beta \\ \alpha &= E^2 - L^2 \\ \beta &= L^2 M - JEL\,. \end{aligned} \tag{3.4}$$

Firstly, when $E^2 \neq L^2$ we get:

$$y = (E^2 - L^2)(\tau - \tau_0)^2 - \frac{L^2 M - JEL}{E^2 - L^2}\,. \tag{3.5}$$

Only for $E^2 > L^2$ does the geodesic reach infinity. When $E^2 < L^2$, the geodesic has a finite range, and reaches a maximum at $y = \frac{L^2 M - JEL}{L^2 - E^2}$. When $E^2 = L^2$, we obtain:

$$y = 2\sqrt{L^2 M - JEL}(\tau - \tau_0)\,. \tag{3.6}$$

Since $E = \pm L$, we have that $L^2 M - JEL \geq 0$. Thus, the geodesic can stretch from infinity to the black hole. In summary, a massless particle can escape the black hole when it has sufficient energy [22].

When $E^2 = L^2$ and $L^2 M = JEL$ (i.e. $J = \pm M$), we encounter a special phenomenon (associated to supersymmetry). We have the solution

$$y = y_0\,. \tag{3.7}$$

The lightray then stays at fixed and arbitrary distance from the black hole [22].

3.2 Timelike geodesics

One can prove that $\alpha^2 + 4m^2\beta \geq 0$ for timelike geodesics (using $M \geq |J|$). The solution to the radial equation for timelike geodesics is:

$$y = \frac{\alpha}{2m^2} + \frac{\sqrt{\alpha^2 + 4m^2\beta}}{2m^2} \sin 2m(\tau - \tau_0)\,. \tag{3.8}$$

Only when $M = \pm J$ and $E = \pm L$ do we obtain the special case $\alpha^2 + 4m^2\beta = 0$. Then the massive particle resides precisely at the horizon of the black hole. Otherwise, the geodesic stretches over a finite interval in the radial direction. The particle crosses the outer horizon at some point, and hits the singularity when $\beta < 0$. Otherwise, it oscillates from inside the inner horizon to outside the outer horizon.

3.3 Spacelike geodesics

Here we depart from a review of [22]. We distinguish two main cases for spacelike geodesics. When $-4m^2\beta > \alpha^2$ we obtain the solution:

$$y = \frac{\alpha}{2m^2} + \sqrt{-\frac{1}{4m^4}(\alpha^2 + 4m^2\beta)} \sinh \sqrt{-4m^2}(\tau - \tau_0)\,. \tag{3.9}$$

The geodesic can stretch from infinity, and will hit the black hole singularity. When $\alpha^2 + 4m^2\beta > 0$, we obtain:

$$y = \frac{\alpha}{2m^2} + \sqrt{\frac{1}{4m^4}(\alpha^2 + 4m^2\beta)} \cosh \sqrt{-4m^2}(\tau - \tau_0)\,. \tag{3.10}$$

Now the geodesic has a radius of nearest approach.

The distinction between the two cases is associated to critical masses (or energy and angular momentum), for a fixed background (M and J). We have that $\alpha^2 + 4m^2\beta = [(E+L)^2 + (M+J)m^2][(E-L)^2 + (M-J)m^2]$. Thus, for fixed M, J and m^2, there is a critical difference between energy and angular momentum for which the second solution applies. Then the solution has a minimal radius $y_{\mathrm{min}} = \frac{\alpha}{2m^2} + \sqrt{\frac{\alpha^2+4m^2\beta}{4m^2}}$. We will discuss this type of solution in a bit more detail for a particular case later.

In the critical case, we have either $(E+L)^2+(M+J)m^2 = 0$ or $(E-L)^2+(M-J)m^2 = 0$. Then the solution to the radial equation is $y = \frac{\alpha}{2m^2} + ce^{\sqrt{-4m^2}(\tau-\tau_0)}$, which shows either a stationary solution, or one that stretches to infinity.

A special case: the vacuum black hole. It is no problem to specialize our formulas to the case of the vacuum black hole (with $M = 0 = J$), on which we concentrate later. We briefly summarize the results.

The null geodesics have $\beta = 0$. Only when $E^2 \geq L^2$ do we have a geodesic solution. When $E^2 = L^2$ it is the radially stationary solution. When $E^2 > L^2$ the solution is given by $r = \sqrt{\alpha}(\tau - \tau_o)$.

For timelike geodesics we have for $E^2 > L^2$ that $r = \sqrt{\frac{E^2-L^2}{m^2}} \cos m(\tau - \tau_0)$, namely a particle oscillating near the singularity, and for $E^2 = L^2$ there is only the solution where the particle sits at $y = 0$.

For spacelike geodesics we find

$$r = \sqrt{\frac{L^2 - E^2}{-m^2}} \cosh \sqrt{-m^2}(\tau - \tau_0) \qquad \text{for} \quad E^2 < L^2$$

which is a solution with a minimal radius, and

$$r = \sqrt{\frac{E^2 - L^2}{-m^2}} \sinh \sqrt{-m^2}(\tau - \tau_0) \qquad \text{for} \quad E^2 > L^2 \,,$$

which hits the radial origin at $\tau = \tau_0$.

3.4 Discussion

We gained two things from the extension of the nice analysis in [22]. On the one hand, we have used formulas that are easily used in the case of the vacuum black hole. That is a fairly trivial extension. On the other hand, we have seen that when a tachyonic particle has sufficient angular momentum, it can stay out of the grasp of the black hole. That leads us to suspect that there may exist long strings that are ordinary massive strings, that stay out of reach of the black hole. Indeed, when we compare to AdS_3 [11], this happens because the long string never shrinks to zero size when it carries angular momentum. It will remain fat when it is spinning sufficiently fast. Thus it may be able to keep away from the event horizon of the black hole. The special case of the vacuum black hole and the distinction between the cases where the energy is larger or smaller than the angular momentum will resurface in later sections.

4. Scalar fields

In this section we review the analysis of the spectrum for scalar fields in the BTZ black hole background (see e.g. [25, 26, 27]). We will not push the spectral analysis to the end to review Hawking radiation, quasinormal modes, etc. Our motivations to include the abbreviated review are the following. Firstly, we point out how to obtain the eigenfunctions of the laplacian for the extremal black hole and vacuum black hole background from the general black hole background by a limiting procedure. This unifies some analyses in the literature and lays bare a technical connection to results in special function theory which may turn out to be useful. Secondly, our analysis will give us the opportunity to strengthen the connection of these backgrounds to $\mathrm{SL}(2,\mathbb{R})$ group theory later.

Indeed, the global structure of the generic BTZ black hole differs from the global structure of the extremal or the vacuum BTZ black hole [6, 28]. That fact reflects in a preferred parametrisation of the $\mathrm{Sl}(2,\mathbb{R})$ group manifold when we will try to formulate the CFT on the BTZ black hole as an $\mathrm{Sl}(2,\mathbb{R})$ orbifold (see e.g. [28]). That choice of parametrisation in turn reflects a convenient choice of basis for $\mathrm{Sl}(2,\mathbb{R})$ representations, which then give rise to different expression for the matrix elements (i.e. kernels) of the representations in terms of special functions. We find a foreshadowing of these facts in the following functional analysis, and will strengthen the connection in section 5.

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4.1 General case

For a scalar field in a generic BTZ black hole background, the Klein-Gordon equation $\Delta\Phi = m^2\Phi(r,\phi,t)$ becomes, after the separation of variables $\Phi = R(r)e^{-iEt}e^{iL\phi}$:

$$\partial_r\partial_r R + f^{-2}\frac{1}{r}\partial_r\left(rf^2\right)\partial_r R + f^{-4}\left(E^2 - \frac{JEL}{r^2} + \left(\frac{M}{r^2} - \frac{1}{l^2}\right)L^2\right)R = 4f^{-2}m^2 R\,, \quad (4.1)$$

where $f^2 = r^2 - M + \frac{J^2}{4r^2}$. It is convenient to define a new variable $z = \frac{r_+^2 - r_-^2}{r^2 - r_-^2}$ and a function $R = (1-z)^\alpha g(z)$ in terms of which the differential equation reads:

$$(z(1-z)\partial_z\partial_z - (2\alpha+1)z\partial_z + A_1 + B_1)g = \frac{m^2}{z}g \quad (4.2)$$

where we made use of the quantities:

$$\begin{aligned} A_1 &= \frac{1}{4}\left(\frac{Er^+ - Lr_-}{r_+^2 - r_-^2}\right)^2 \\ &= k_+^2 \\ &= -\alpha^2 \\ B_1 &= \frac{1}{4}\left(\frac{Er^- - Lr_+}{r_+^2 - r_-^2}\right)^2 \\ &= -k_-^2\,. \end{aligned} \quad (4.3)$$

Next, by standard manipulations, we obtain the usual form of the differential equation for hypergeometric functions. Indeed, defining $g = z^\beta X$, $\alpha = ik_+$ and choosing $\beta(\beta-1) = m^2$, we obtain:

$$(z(1-z)\partial_z\partial_z + (2\beta - (2\alpha + 2\beta + 1)z)\partial_z + (-\beta^2 - 2\alpha\beta + A_1 + B_1))X = 0\,. \quad (4.4)$$

In terms of the (standard) parameters

$$\begin{aligned} a + b &= 2(\alpha + \beta) \\ ab &= \beta^2 + 2\alpha\beta - (A_1 + B_1) \\ c &= 2\beta\,, \end{aligned} \quad (4.5)$$

the solutions to the hypergeometric equations are $F(a,b,c,z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c, z)$. We will choose the parameters a and b to be

$$\begin{aligned} a &= \alpha - ik_- + \beta = \frac{i(E+L)}{2(r_+ + r_-)} + \beta \\ b &= \alpha + ik_+ + \beta = \frac{i(E-L)}{2(r_+ - r_-)} + \beta\,. \end{aligned}$$

As usual, the physical problem under consideration, whether it be Hawking radiation, quasinormal modes or otherwise, determines the relevant combination of solutions to the differential equations, by fixing the boundary conditions. See e.g. [25, 26, 27].

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4.2 Extremal limits

As advertised, the differing global structure of the extremal and massless BTZ black hole are reflected in the analysis of the spectrum of a scalar field in these backgrounds. We briefly show how this connects to limiting procedures in the theory of special functions.

Extremal black hole. Firstly, we concentrate on the extremal black hole with non-zero mass. The inner and outer horizon coincide for this black hole. We can approach the extremal black hole within the BTZ configuration space by letting the outer horizon r_+ tend to the inner horizon r_- from above, keeping all other quantities fixed. The parameter

$$b = \frac{i(E-L)}{2(r_+ - r_-)} + \beta$$

of the hypergeometric function becomes infinite in this limit, the $z = \frac{r_+^2 - r_-^2}{r^2 - r_-^2}$-coordinate tends to zero, while $y = zb$ remains finite. When implementing this in the differential equation (4.4) for the hypergeometric function , we obtain the equation for the confluent hypergeometric function [29]:

$$y\partial_y \partial_y S + (c-y)\partial_y S - aS = 0\,. \tag{4.6}$$

The solutions can be written in terms of Whittaker's functions [29].

Vacuum black hole. To obtain the vacuum black hole, we need to take a further limit, along the space of supersymmetric black holes. When r_+ tends to zero (along a line of supersymmetric black hole backgrounds) we have that the parameter $a = \frac{i(E+L)}{4r_+} + \beta$ tends to infinity in (4.6). After defining the variable $x = ay$ which remains finite in this limit, we obtain the radial equation for the vacuum black hole:

$$x\partial_x \partial_x T + c\partial_x T - T = 0\,. \tag{4.7}$$

After further redefining $T = x^{\frac{1}{2}(1-c)}U$ and $w = 2x^{\frac{1}{2}}$ we find the standard form of the Bessel equation:

$$w^2 \partial_w \partial_w U + w\partial_w U - (w^2 + (1-c)^2)U = 0\,. \tag{4.8}$$

Thus the scalar wave function can be written in terms of Bessel functions with index $\nu = 2\tau + 1$ (where $\tau = -\beta$).

For this case, which will be of particular interest to us later, we give a little more detail. In terms of the original (r, ϕ, t) coordinates, we have, for $E^2 > L^2$:

$$\Phi = e^{-iEt} e^{iL\phi} \frac{1}{r} \left(c_1 J_{2\tau+1}\left(\frac{\sqrt{E^2 - L^2}}{r}\right) + c_2 J_{-(2\tau+1)}\left(\frac{\sqrt{E^2 - L^2}}{r}\right)\right) \tag{4.9}$$

and the solution regular in the interior for $E^2 < L^2$ is:

$$\Phi = c e^{-iEt} e^{iL\phi} \frac{1}{r} K_{2\tau+1}\left(\frac{\sqrt{L^2 - E^2}}{r}\right)\,. \tag{4.10}$$

Without going into the details of important special cases (for instance discussed in the context of AdS_3 in [45, 9]), we remark that for τ real, we have for $E^2 > L^2$ an oscillating solution that is localized near the origin. For $2\tau + 1$ purely imaginary (i.e. $m^2 \leq -1/4$), we find solutions that travel to the boundary. In the timelike case, we find incoming and outgoing waves, while in the second case these waves are related by demanding regularity. Thus, the picture that emerges is consistent with the analysis of geodesics presented at the end of section 3, where we found timelike geodesics concentrated near the singularity, and spacelike geodesics that could travel to the boundary.[2]

4.3 Summary

The limiting procedures in special function theory that relate the hypergeometric functions to confluent hypergeometric functions and Bessel functions have a natural interpretation as limiting procedures for scalar wave functions in generic, extremal, or massless BTZ black hole backgrounds. This is a foreshadowing of the global picture we paint in the next section.

5. Strings and BTZ black holes

In this section we will assemble some general remarks on strings in BTZ black hole backgrounds. We stress the importance of the group theoretic approach, which should enable us to make use of results of WZW-models, generalized to non-compact groups. We recall some facts on the analysis of strings in AdS_3 backgrounds, and their connection to $\mathrm{Sl}(2,\mathbb{R})$ representation theory. Next, we discuss the different group parametrisations suited for a generic, an extremal and a vacuum BTZ black hole background, and connect these parametrisations to the special function theory of the previous section. Then we point out the generic occurence of wound strings in these backgrounds, classically. We point out the close analogy of the generic BTZ black hole to a recently studied stringy cosmological background, and point out a dilemma.

This section could serve as a generic framework for discussing strings in BTZ backgrounds algebraically. In the next section, we will restrict attention to the case that interests us most in this paper, the vacuum black hole.

5.1 Strings on AdS_3

In this subsection, we review some features of strings on AdS_3 that will find a generalisation in strings in BTZ backgrounds. We follow the systematic treatment of [11] to which we the reader for a lot more details. The manifold AdS_3 (with no closed timelike loops) is the covering space of the group manifold $\mathrm{Sl}(2,\mathbb{R})$. To study strings in AdS_3, it is useful to study the $\mathrm{Sl}(2,\mathbb{R})$ Wess-Zumino-Witten model, where we parametrize the group elements as

$$g = e^{\frac{i}{2}(t+\phi)\sigma^2} e^{\rho\sigma^3} e^{\frac{i}{2}(t-\phi)\sigma^2} \, . \tag{5.1}$$

[2]Our choice of values for τ will find justification in the next section, from the representation theory of $\mathrm{Sl}(2,\mathbb{R})$.

One needs to be careful in defining the configuration space, though, to ensure that the strings do live on the covering of the group manifold $\mathrm{Sl}(2,\mathbb{R})$ — we unwrap the time coordinate t.

A strong motivation for the choice of parametrisation (5.1) is that it allows for a straightforward diagonalisation of the energy and angular momentum, by a choice of basis in $\mathrm{Sl}(2,\mathbb{R})$ representations that diagonalize J_0^3 and $\bar{J}_0^3$, which generate time translation and rotations [11]. Thus, by our choice of parametrisation, it is easy to diagonalize the action of an elliptic subgroup of $\mathrm{Sl}(2,\mathbb{R})$ (see [34, p. 377]).

Excitations in AdS_3 string theory fall into representations of (the affine extension of) the covering group of $\mathrm{Sl}(2,\mathbb{R})$. The unitary representations of the covering are enumerated in the appendix. The demand of unitarity yields a first restriction on the possible masses for particles in AdS_3 spaces. These restrictions consequently lead to a restriction on the string Hilbert space.

Within the set of unitary representations of the covering group of $\mathrm{Sl}(2,\mathbb{R})$, we further restrict ourselves to two types of unitary irreducible representations, namely the principle discrete and the principle continuous representations. (We thereby disregard the principle complementary series, which do not arise as quadratically integrable particle wavefunctions on AdS_3.)

Due to the $(\widetilde{\mathrm{Sl}(2,\mathbb{R})} \times \widetilde{\mathrm{Sl}(2,\mathbb{R})})/\mathbb{Z}_2$ background isometrygroup, we know the particle excitations will form representations of $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$. More precisely, the particle wavefunctions will be tensor representations of the $(\widetilde{\mathrm{Sl}(2,\mathbb{R})} \times \widetilde{\mathrm{Sl}(2,\mathbb{R})})/\mathbb{Z}_2$ symmetry-group, which can be represented as matrix elements of the $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ representation R_χ labelled by $\chi = (\tau, \epsilon)$, with quadratic Casimir $c_2 = -\tau(\tau+1)$ [34].[3] The mass squared of the particle is $m^2 = \tau(\tau+1)$.

The principle continuous series of representations has $\tau = -1/2 + is$ such that $m^2 = -1/4 - s^2$, in violation of the Breitenlohner-Freedman bound. This indicates that these are unstable modes (in a spacetime that is asymptotically AdS). The representations are associated to spacelike geodesics in the classical picture. We moreover have the principle discrete series with $m^2 \geq -1/4$, such that (most of) these correspond to ordinary particles in AdS_3.

Spectral flow. It was further argued in [11, 12, 13] that the solutions obtained by acting with a non-trivial automorphism on a given geodesics solution to the equations of motion, yields winding strings [30, 31] that are crucial for the determination of the spectrum for strings on AdS_3 backgrounds. We refer to [11] for an extensive discussion.

5.2 Black hole

Similarly, we want to make some exploratory remarks on string theory on BTZ black hole backgrounds. We will show that different regions of spacetime require different parametrisations of the $\mathrm{Sl}(2,\mathbb{R})$ group elements. We link this picture to recent explorations of cosmological models in string theory in passing.

[3] The parameter $\epsilon \in \{0, 1/2\}$ labels whether the center is represented non-trivially. It will not play a crucial role in the following.

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Generic black hole For the case of the generic black hole, we assemble a few perhaps well-known facts. First of all we note that we can write every $\mathrm{Sl}(2,\mathbb{R})$ group element with all non-zero matrix elements as follows:[4]

$$g = e^{\frac{u}{2}\sigma^3}(-1)^{\epsilon_1}.(i\sigma^2)^{\epsilon_2}.p.e^{\frac{v}{2}\sigma^3} \tag{5.2}$$

where $\epsilon_{1,2} \in \{0,1\}$ and the two by two matrix p takes one of the following two forms:

$$p_1 = e^{\rho\sigma_1} \tag{5.3}$$
$$p_2 = e^{i\rho\sigma_2}\,. \tag{5.4}$$

Next we note that we can divide the $\mathrm{Sl}(2,\mathbb{R})$ group manifold into 3 large regions (each consisting of 4 smaller regions). (See e.g. [32] for a discussion in a physical context.) When we take the convention that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the three regions are characterized by the signs of ad and bc. Region I, where $ad > 0$ and $bc > 0$ can be shown [6] to correspond to the region outside the outer horizon, region II ($ad > 0$ and $bc < 0$) to the region between the inner and the outer horizon, and region III ($ad < 0$ and $bc < 0$) to a region inside the inner horizon. Each of the twelve regions can be parametrized in terms of a group element written as in (5.2). The precise correspondence can be found in [32].

As an example, consider part of the region $ad > 0$ and $bc > 0$. We can parametrize the $\mathrm{Sl}(2,\mathbb{R})$ group element as: $g = e^{\frac{u}{2}\sigma_3}p_1 e^{\frac{v}{2}\sigma_3}$. From this we can easily find the metric, and we can transform to the usual Schwarschild type coordinates via the coordinate transformation:

$$\begin{aligned} \cosh^2\rho &= \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \\ u &= (r_+ - r_-)(t + \phi) \\ v &= (r_+ + r_-)(\phi - t)\,. \end{aligned} \tag{5.5}$$

Clearly, this describes a region of spacetime beyond the outer horizon. An important point is that the parametrisation (5.2) is suited for the diagonalisation of a hyperbolic subgroup of $\mathrm{Sl}(2,\mathbb{R})$ [26, 40]. For this background the hyperbolic parametrisation naturally leads to diagonalisation of the energy and angular momentum, since these are associated to hyperbolic generators in this background. The matrix elements in that basis can be written down in terms of hypergeometric functions ([34, section 7.2]). This squares nicely with the functional analysis in section 4 and shows that an algebraic analysis close to the two-dimensional treatment in [2] of black hole scattering, Hawking radiation, etc, can be repeated in this context.

Brief remark. We point out that there is some unresolved tension between two reasonable points of view in the literature. To avoid closed timelike curves, the authors of [6] cut out part of region III from the spacetime, and it was moreover argued in a general relativistic context that matter couplings (and a curvature singularity that could arise from matter at $r = 0$) would force one to do so (see [6, section V]).

[4] A separate treatment of the group elements with a zero entry is necessary.

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In the stringy and algebraic context of a coset WZW conformal field theory the authors of [32] kept the region III (compare [6, figure 2, region III] to [32, figure 1]) which seemed natural from a group theoretic perspective. They pointed out that the physics of closed timelike curves needs further study in their stringy cosmological backgrounds.

Although the contexts of application of the $\mathrm{Sl}(2,\mathbb{R})$ group theory are different, it seems to us that there is some tension between these two approaches. We will not try to resolve it here, but believe it is crucial to compare these backgrounds in more detail on this particular point (and decide on whether it makes sense to add the regions of spacetime that contain closed timelike curves within a string theoretic context).

5.3 Extremal black holes

A similar story can be told for extremal black holes (see also [19, 33]), where a mixed basis parametrisation leads to straightforward diagonalisation of an parabolic and a hyperbolic subgroup, corresponding to energy and angular momentum generators. The kernels can in this case be written in terms of Whittaker functions ([34, section 7.7.3 and 3.5.7]). For the vacuum black hole, two parabolic subgroups are to be diagonalised on physical grounds. Then the kernels (i.e. matrix elements) may be written in terms of Bessel functions ([34, section 7.6]) in full agreement with the analysis in section 4.

5.4 Winding strings

After stressing the physical logic (i.e. diagonalisation of conserved quantities) behind the global parametrisations of the $\mathrm{Sl}(2,\mathbb{R})$ group manifold, it becomes straightforward to discuss the particular classical solutions corresponding to strings winding the BTZ black hole. The propagation of strings in black hole backgrounds will be described by an (orbifolded) $\mathrm{Sl}(2,\mathbb{R})$ WZW model (where different parametrisations are appropriate according to the background and particular patch of spacetime) (see also [28, 40]). The classical solutions to the string equations of motion for a WZW model are given by a product of left and right movers as $g = g^+(x^+)g^-(x^-)$ (where $x^\pm = \tau \pm \sigma$). After our preliminary investigations, it is easy to state the generalisation of the solution to the equations of motion that will provide us with the winding strings in black hole backgrounds. We can start from a solution to the equations of motion that represents a pointlike string moving on the geodesics discussed in section 3. Then we make use of the following general technique.

The closed string only has to close up to the action of an element of the orbifold group, which implements the periodicity in the angular coordinate ϕ. Thus we have that the string (given by an embedding in the group manifold) closes as $g(\sigma+2\pi) = (h_l)^n g^+(x^+ + 2\pi)g^-(x^- - 2\pi)(h_r)^n$, where $h = (h_l, h_r)$ is the generator of the infinite discrete orbifold group (which is a subgroup of $\mathrm{Sl}(2,\mathbb{R})_L \times \mathrm{Sl}(2,\mathbb{R})_R$).

A typical solution in the n-twisted sector would be $g^+(x^+ + 2\pi) = (h_l)^n g^+(x^+)$ and $g^-(x^- - 2\pi) = g^-(x^-)(h_r)^n$ where n labels the twisted (winding) sectors of the orbifold. The modification from the solutions that represent pointlike geodesic motion are easily found from the appropriate parametrisations of the group elements in all backgrounds and patches. (This is straightforward because we made sure that the parametrisations are such that the energy and angular momentum are naturally diagonal. Formally speaking, we made sure that the outer factors of the group element decomposition always contain the angular and time coordinate.) Thus, we constructed the classical winding string solution in all generality. We refrain from enumerating in great length all cases — in the next section we will have the opportunity to study one particular case in detail.

Although the above reasoning is sufficient to establish the existence of these classical solutions, we included an explicit parametrisation and check of the solutions in appendix A, in an alternative formalism.

6. Vacuum black hole

Previous sections were concerned with coming to grips with fitting the mathematical results on $\mathrm{Sl}(2,\mathbb{R})$ group theory into the physical framework of BTZ black holes, while obtaining some more insight into the geodesic structure of BTZ black holes, the global structure of the spacetime and of $\mathrm{Sl}(2,\mathbb{R})$, and the relation to results in the theory of special functions. Apart from the new physics that we uncovered by a careful review and extension, our analysis also prepared us for some of the new phenomena we will encounter in a more detailed study of the particular case of the vacuum black hole.

The metric for the vacuum black hole with $M = 0 = J$ is given by

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2 \tag{6.1}$$

with $\phi \in [0, 2\pi[$ (and we put $l = 1$ again from now on). Following the general logic, we define the group element[5]

$$g = e^{v_l L^-} e^{\rho\sigma^3}.s.e^{-v_r L^-} \tag{6.2}$$

where σ^i are the Pauli matrices and $L^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$. The parametrisation is related to the previous coordinates by the transformation

$$\begin{aligned} v_l &= \phi + t & (6.3)\\ v_r &= \phi - t & (6.4)\\ e^\rho &= r & (6.5) \end{aligned}$$

In terms of these coordinates the metric reads:

$$ds^2 = e^{2\rho} dv_l dv_r + d\rho^2 \,, \tag{6.6}$$

which is the invariant metric on the group manifold. The global angular identification in these coordinates is $(v_l, v_r, \rho) \equiv (v_l + 2\pi, v_r + 2\pi, \rho)$, which is generated by $(h_l, h_r) = (e^{2\pi L^-}, e^{-2\pi L^-})$.

[5]To be a little more precise, every $\mathrm{Sl}(2,\mathbb{R})$ group element can be parametrized as follows. Either as $g = e^{v_l L^-} e^{\rho\sigma^3}.(i\sigma^2).e^{-v_r L^-}(-1)^\nu$ or, for group elements with a zero: $g = e^{v_l L^-} e^{\rho\sigma^3}(-1)^\nu$. We mod out by -1 (and put $\nu = 0$). The group elements with a zero should be thought of as located at the black hole singularity. We do not discuss the subtleties associated to this patch here.

We can define the currents as $J^i_L = kTr(L^i\partial_+ gg^{-1})$ for the leftmovers, and $J^i_R = kTr(L^i g^{-1}\partial_- g)$ for the rightmovers, where $L^3 = \frac{i}{2}\sigma^3$. The leftmoving currents can be computed to be:

$$\begin{aligned} J^3_L &= ik(\partial_+\rho - e^{2\rho}v_l\partial_+ v_r) \\ J^-_L &= ke^{2\rho}\partial_+ v_r \\ J^+_L &= k(\partial_+ v_l + 2v_l\partial_+\rho - v_l^2 e^{2\rho}\partial_+ v_r) \end{aligned} \tag{6.7}$$

and the rightmoving ones are:

$$\begin{aligned} J^3_R &= -ik(\partial_-\rho - e^{2\rho}v_r\partial_- v_l) \\ J^-_R &= -ke^{2\rho}\partial_- v_l \\ J^+_R &= -k(\partial_- v_r + 2v_r\partial_-\rho - v_r^2 e^{2\rho}\partial_- v_l)\,, \end{aligned} \tag{6.8}$$

where we used the cylindrical worldsheet coordinates $x^\pm = \tau \pm \sigma$.

6.1 Winding sectors

We obtain the twisted sector of the orbifolded theory by generalized spectral flow from the untwisted sector. As explained before (in subsection 5.4), we can add winding to a classical (e.g. geodesic) solution $\tilde{g} = \tilde{g}^+\tilde{g}^-$ by the substitution:

$$\begin{aligned} g^+ &= e^{nx^+L^-}\tilde{g}^+ \\ g^- &= \tilde{g}^- e^{nx^-L^-}\,. \end{aligned} \tag{6.9}$$

Clearly, this generates a new solution to the equations of motion. Moreover, if the original solution belonged to the untwisted sector, than the new solution belongs to the sector twisted by n units, since $g(\sigma+2\pi) = g^+(x^+ + 2\pi)g^-(x^- - 2\pi) = e^{2\pi nL^-}g(\sigma)e^{-2\pi nL^-}$.

The action of this operation on the currents is:

$$\begin{aligned} J^-_L &= \tilde{J}^-_L \\ J^3_L &= \tilde{J}^3_L - inx^+\tilde{J}^-_L \\ J^+_L &= \tilde{J}^+_L - (nx^+)^2\tilde{J}^-_L - 2inx^+\tilde{J}^3_L + kn\,, \end{aligned} \tag{6.10}$$

and similarly for the rightmovers. The action on the energy momentum tensor for right- and leftmovers is given by:

$$\begin{aligned} T_{++} &= \tilde{T}_{++} + n\tilde{J}^-_L + T^{\text{other}}_{++} \\ T_{--} &= \tilde{T}_{--} + n\tilde{J}^-_R + T^{\text{other}}_{--} \end{aligned} \tag{6.11}$$

where we used the conventions that T^{other}_{++} corresponds to the part of the string wordlsheet CFT that describes the seven directions transverse to the BTZ black hole. (Our conventions are similar to the ones in [11].) That implies the following transformation rule for the worldheet zeromodes of the stress tensor:

$$\begin{aligned} L_0 &= \tilde{L}_0 + n\tilde{J}^-_{L,0} + L^{\text{other}}_0 \\ \bar{L}_0 &= \tilde{\bar{L}}_0 + n\tilde{J}^-_{R,0} + L^{\text{other}}_0\,. \end{aligned} \tag{6.12}$$

Now, the time translation generator and angular momentum generator in this background are given by:

$$E = -(J^-_{L,0} + J^-_{R,0})$$
$$L = J^-_{L,0} - J^-_{R,0}\,. \tag{6.13}$$

Note that the transformation properties of the currents imply that the energy of the strings in the twisted sectors are degenerate (but the constraint equations will require different internal conformal weights).

When we assume that $\tilde{T}_{++} = -km^2$ (as for a geodesic) and $T^{\rm other}_{++} = h$ for the internal conformal field theory, and similarly for the rightmovers, the zeromode constraint equations in the twisted sectors read:

$$-km^2 + \frac{n}{2}(L - E) + h = 0$$
$$-km^2 + \frac{n}{2}(-L - E) + \bar{h} = 0\,. \tag{6.14}$$

Thus, the internal conformal weights are related to the spacetime angular momentum by $\bar{h} - h = nL$. The spacetime angular momentum is quantized, i.e. $L \in \mathbb{Z}$.

For $n = 0$, the only solutions to the constraints (6.14) are associated to timelike geodesics. (We assume that the conformal weights h and $\bar{h}$ associated to the internal conformal field theory are positive.) But now we notice that in the twisted sectors, we can find solutions to the constraints that arise from spacelike geodesics for $E^2 > L^2$ and n positive. The fact that the spacelike geodesics are promoted to viable solutions to the constraints in the twisted sector is similar to the picture obtained in [11].

7. Quantum CFT description

7.1 Spectrum and ghosts

Now we come to an important point. The classical current algebra (6.10) in the twisted sector is the classical counterpart of the current algebra that was introduced in [37] (p.4 (3.2))[6] and [38]. There, a modification of the currents was formally introduced to obtain a ghost free spectrum for string theory on $\mathrm{Sl}(2,\mathbb{R})$ backgrounds.[7] We see that these currents arise natural in our approach when analysing winding strings in the vacuum black hole background.

We can then apply the results of [37] in this context. Specifically, we will find physical states in our quantum Hilbert space that satisfy (compare also to the analogous treatment in AdS_3 [11]):

$$L_0 - 1|\tilde{\tau}, h, \tilde{N}, j_0^-\rangle = -\frac{\tilde{\tau}(\tilde{\tau}+1)}{k-2} + \tilde{N} + h - 1 + nj_0^-|\tilde{\tau}, h, \tilde{N}, j_0^-\rangle = 0$$
$$\bar{L}_0 - 1|\tilde{\tau}, h, \tilde{\bar{N}}, \bar{j}_0^-\rangle = -\frac{\tilde{\tau}(\tilde{\tau}+1)}{k-2} + \tilde{\bar{N}} + \bar{h} - 1 + n\bar{j}_0^-|\tilde{\tau}, h, \tilde{\bar{N}}, \bar{j}_0^-\rangle = 0\,, \tag{7.1}$$

[6]Note that [37] works with a worldsheet coordinate on the plane were we worked on the cylinder in the previous section.

[7]A formal connection to a generic BTZ background was proposed in [41].

where the tilded quantities $\tilde{\tau}$ and $\tilde{N}$ refer to Casimir and oscillation number before twisting. The quantity j_0^- is the eigenvalue of the J_0^- operator. The quantisation of angular momentum, equivalent to periodicity in the spacetime coordinate ϕ, is now the natural interpretation of the "monodromy" projection in [37]. It was proven in [37] that restricting to continuous representations gives a ghost free spectrum. We will not analyse in detail how the discrete representations fit into the picture, although we expect them to occur in the spectrum, as in the AdS_3 background. One obstruction to a completion of the spectral analysis is the fact that the twisted sectors are not easily expressed in terms of a representation of the original current algebra, in contrast with the AdS_3 case.

Now, in [39] there is a suggestion that the form of the current algebra is linked to winding strings in an AdS_3 background, resulting in some unresolved tension with the works [7, 8, 9]. We resolved the tension by noticing that the logarithmic cuts (in terms of a planar worldsheet coordinate), and the winding strings, are properly interpreted as being apart of the string conformal field theory in the background of the massless BTZ black hole.

7.2 Spacetime Virasoro algebra

A second issue we want to address in the quantum theory, is the construction of the spacetime Virasoro algebra in terms of operators in the worldsheet conformal field theory. For string theory on AdS_3 backgrounds, the construction was obtained in [7, 8, 9]. Since the BTZ black holes are also asymptotically AdS_3, we expect on the basis of the pioneering work [36] to find a realisation of the Virasoro algebra acting on the space of physical states as well. To obtain a construction of the spacetime Virasoro generators, we study the conformal field theory in euclidean signature in target space and on the worldsheet (parametrized by the planar coordinate z).

Note first of all that the conformal field theory description of the euclidean AdS_3, is globally given by a lagrangian:

$$L = k(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\eta\partial\bar{\eta})\,, \tag{7.2}$$

where η takes values on the plane. By contrast, the euclidean version of the BTZ vacuum black hole is described by the CFT:

$$L = k(\partial\rho\bar{\partial}\rho + e^{2\rho}\bar{\partial}\gamma\partial\bar{\gamma})\,, \tag{7.3}$$

where $\gamma \simeq \gamma + 2\pi$ takes values on the cylinder. We want to discuss the latter CFT.

As described for instance in [7] in some detail, one can approximate the above conformal field theory near the boundary by a free field theory. The lagrangian

$$L = \partial\rho\bar{\partial}\rho - \frac{2}{\alpha_+}R^{(2)}\rho - \beta\bar{\partial}\gamma - \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\rho}\,, \tag{7.4}$$

where $\alpha_+^2 = 2k - 4$, is equivalent to the previous one, and we can compute correlation functions that are dominated by the large ρ region by ignoring the last term.

In the free field approximation to the conformal field theory, we can realize the Sl(2) current algebra:

$$
\begin{aligned}
J^3(z)J^{\pm}(w) &\simeq \frac{\pm J^{\pm}(w)}{z-w} \\
J^3(z)J^3(w) &\simeq -\frac{k}{2}\frac{1}{(z-w)^2} \\
J^-(z)J^+(w) &\simeq \frac{k}{(z-w)^2}+\frac{2J^3(w)}{z-w}
\end{aligned}
\tag{7.5}
$$

using the free fields:

$$
\begin{aligned}
J^3 &= -\beta\gamma+\frac{\alpha_+}{2}\partial\rho \\
J^+ &= -\beta\gamma^2+\alpha_+\gamma\partial\rho+k\partial\gamma \\
J^- &= -\beta
\end{aligned}
\tag{7.6}
$$

where we use the conventional OPE for the free fields $\rho(z)\rho(w) \simeq -\log(z-w)$ and $\beta(z)\gamma(w) \simeq -\frac{1}{z-w}$.

Since the black hole is asymptotically AdS_3, we expect a realisation of the Virasoro algebra to act on the space of physical states [36]. The boundary of our space is $R^1 \times S^1$, because of the identification on γ. First of all, we note that the spacetime $\mathcal{L}_0$ should measure spacetime energy and angular momentum (see also (6.13)), and is therefore roughly given by:[8]

$$\mathcal{L}_0 \sim \oint dz J^-(z)\,. \tag{7.7}$$

Now, since we expect the conformal weight of the higher generators to be related to their index, the index is expected to be related to the J_0^- charge. We propose to describe the spacetime Virasoro generators in terms of the free fields as follows:

$$\mathcal{L}_n = -i\oint dz\left(J^-+\frac{in\alpha_+}{2}\partial\rho+\frac{k}{4}\partial\gamma\right)e^{in\gamma}(z)\,. \tag{7.8}$$

This expression is also inspired by the form of the classical Virasoro algebra on the cylinder, the periodicity of γ, and the conformal dimensions of the operators in the theory. It can be checked that the coefficients of the first and second term have the right ratio to make sure the integrand is primary. The coefficient of the third term is such that the anomalous term (linear in n) in the Virasoro algebra works out. Indeed, the commutator of these Virasoro operators can be evaluated using the contour argument, and the OPE of the free fields that make up the operators $\mathcal{L}_n$. We obtain the algebra:

$$[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n}\,, \tag{7.9}$$

where $c = 6kw$ and $w = i\oint\partial\gamma$ measures the winding number of the string at infinity. This is a nice black hole counterpart to the construction presented in [7]. It would be interesting

[8]We will restrict to writing formulas for leftmovers only when the rightmovers behave in close analogy.

to understand the analogue of this construction for strings on general BTZ backgrounds. Note that the Virasoro generator $\mathcal{L}_n$ acts on the field γ roughly as $\mathcal{L}_n \sim e^{in\gamma}\partial_\gamma$, as expected from the standard form of the conformal algebra on the cylinder.

Moreover, it should be clear that if a further affine symmetry (with currents $K^a(z)$) is present on the worldsheet, arising from the internal CFT, we can construct a corresponding target space affine algebra in a manner formally similar to [7] by introducing the spacetime operators $T_n^a = \oint dz K^a(z) e^{in\gamma}$. The target space current algebra will have a level equal to the level of the worldsheet current algebra, multiplied by the winding number.

Remark on wound free fields. To describe the winding strings in the quantum picture, we can introduce a new field α such that $\beta \equiv \partial\alpha$ and

$$\alpha(z)\gamma(w) \simeq -\log(z-w)\,. \tag{7.10}$$

We can define the current $J_\gamma = -i\partial\gamma$. We expect the zeromode to describe momentum and winding in the direction γ, which is the direction around which we want to wind the string. The operator $e^{iq\alpha}$ has charge q, and can therefore be used to introduce vertex operators corresponding to winding strings.

We will not attempt to give a complete discussion of vertex operators here, but one should be able to recuperate results of [39] in this context. It would certainly be interesting to revisit the discussion in [39] in the light of our conceptual clarification.

8. Conclusions and future directions

In this paper, we analysed string theory on AdS_3 black hole backgrounds. We argued strongly for the importance of an appropriate parametrisation of the $\mathrm{Sl}(2,\mathbb{R})$ group manifold. Depending on whether one is studying the massless black hole, the extremal black hole, or a generic black hole, one will choose a parametrisation in which two parabolic subgroups are easily diagonalised, a parabolic and a hyperbolic subgroup, or two hyperbolic one parameter subgroups. That choice allows for an algebraic treatment of the eigenfunctions of the laplacian, as well as for a straightforward implementation of the orbifolding of the WZW theory. In the case of the generic BTZ black hole, we pointed out a close analogy to recent work on stringy cosmology [32], and showed the tension between the algebraic and the general relativistic approach.

By analysing geodesic motion, and classical string solutions, we gained intuition for the spectrum of strings on BTZ backgrounds. Next, we focussed on the massless BTZ black hole background, because of its simplicity, supersymmetry, its connection to pure AdS_3, and its relation to recent attempts to study cosmological backgrounds in string theory (see e.g. [32], [46]–[49] for references closest to our work). By taking the global structure of the space properly into account, and using a classical analysis of wound strings, we argued that the proposal for a ghost free spectrum of [38] finds a natural interpretation as pertaining to strings on the massless black hole background. It follows that the results of [39] can be reinterpreted as applying to the AdS/CFT duality for the massless BTZ black hole.

As a first important application of our analysis, we provided the construction of the spacetime Virasoro algebra in the massless black hole background. Thus, we gave another explicit realisation of the spacetime boundary conformal algebra in a quantum theory of gravity with propagating degrees of freedom.

There are many interesting directions for future research. Of particular interest would be the rigorous analysis of the exact spectrum (along the lines of [12, 50]) for strings in the massless black hole background, preferably in a supersymmetric context (see e.g. [51]), such that, after applying the AdS_3/CFT correspondence, it can be compared, via spectral flow in the spacetime boundary conformal field theory, to the spectrum of strings on AdS_3 (see e.g. [52, 53] and references therein). That would provide a nice check of the consistent spectrum for strings on a black hole background. Other directions are the application of the orbifold procedure on the euclidean $\mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2)$ CFT so as to describe strings propagating on the euclidean BTZ black holes [54].

In [13] the scattering amplitude for two long wound strings in AdS_3 was suggested as an interesting computation, because of its interpretation in terms of an S-matrix in an AdS_3 background. It would be even more interesting if the scattering process would yield information on the production rate of black holes in a quantum theory of gravity. In the context of classical three-dimensional gravity, it was shown that the collision of two fast particles in AdS_3 would yield a black hole [55]. (See also e.g. [56, 57].)

We hope to return to some of these issues in the future.

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A. How to lasso a black hole

For a string worldsheet action that reads

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma \left(G_{\mu\nu}\left(\dot{X}^\mu \dot{X}^\nu - X^{\mu\prime} X^{\nu\prime}\right) + B_{\mu\nu}\left(X^{\mu\prime}\dot{X}^\nu - \dot{X}^\mu X^{\nu\prime}\right)\right) \tag{A.1}$$

after gauge fixing, the string equations of motion are:

$$\ddot{X}^\mu - X^{\mu\prime\prime} + \Gamma^\mu{}_{\rho\nu}(\dot{X}^\rho \dot{X}^\nu - X^{\rho\prime} X^{\nu\prime}) + H^\mu{}_{\rho\nu}(X^{\nu\prime}\dot{X}^\rho - \dot{X}^\nu X^{\rho\prime}) \tag{A.2}$$

where

$$\Gamma^\mu{}_{\rho\nu} = \frac{1}{2}G^{\mu\alpha}(G_{\alpha\nu,\rho} + G_{\alpha\rho,\nu} - G_{\rho\nu,\alpha}) \tag{A.3}$$

$$H^\mu{}_{\rho\nu} = \frac{1}{2}G^{\mu\alpha}(B_{\rho\nu,\alpha} - B_{\alpha\nu,\rho} + B_{\alpha\rho,\nu})\,. \tag{A.4}$$

These equations should be supplemented with the constraint equations:

$$\begin{aligned} G_{\mu\nu}X^{\mu\prime}\dot{X}^\nu &= 0 \\ G_{\mu\nu}(\dot{X}^\mu \dot{X}^\nu + X^{\mu\prime}X^{\nu\prime}) &= 0\,. \end{aligned} \tag{A.5}$$

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Note in particular that a σ-independent solution for classical strings is a geodesic in space-time. (We made us of that fact in section 3.)

Winding strings. As remarked in section 5.4, in the BTZ background, we can obtain winding strings by starting from geodesics, and adding a winding in ϕ, which is accompanied by a stretch of the string in the time direction t. The new classical solution is given by:

$$\begin{aligned} r &= r_0(\tau) \\ \phi &= \phi_0(\tau) + n\sigma \\ t &= t_0(\tau) + n\tau\,, \end{aligned} \tag{A.6}$$

where the original solution (labelled by $'0'$) represents a geodesic. It is then easy to see that the new terms in the equations of motion (A.2) are either linear or quadratic in n. The term linear in n is proportional to $(\Gamma^{\mu}{}_{\nu t} + H^{\mu}{}_{\nu\phi})\dot{X}^{\nu}$ (where μ, ν are generic). This can be proven to vanish for the BTZ background using the metric, the H-field ($H_{r\phi t} = r$), and the Christoffel connection:

$$\begin{aligned} \Gamma^{r}{}_{tt} &= rf^2 \\ \Gamma^{t}{}_{tr} &= rf^{-2} \\ \Gamma^{\phi}{}_{tr} &= \frac{J}{2r}f^{-2} \\ \Gamma^{r}{}_{rr} &= \frac{J^2 - 4r^4}{4r^3}f^{-2} \\ \Gamma^{t}{}_{\phi r} &= -\frac{J}{2r}f^{-2} \\ \Gamma^{\phi}{}_{r\phi} &= \frac{r^2 - M}{r}f^{-2} \\ \Gamma^{r}{}_{\phi\phi} &= -rf^2\,, \end{aligned} \tag{A.7}$$

where $f^2 = r^2 - M + \frac{J^2}{4r^2}$. Similarly, the term quadratic in n, proportional to $\Gamma^{\mu}{}_{tt} - \Gamma^{\mu}{}_{\phi\phi} - 2H^{\mu}{}_{\phi t}$ vanishes in the WZW background. The constraint equations can be satisfied by making use of the internal degrees of freedom (i.e. non-trivial dynamics in the other seven dimensions of spacetime). The analysis in this appendix is similar in spirit to the approach of e.g. [42, 43, 44].

B. $\mathrm{Sl}(2,\mathbb{R})$ representation theory

We are interested in representations of the Lie algebra of generators of the group $\mathrm{SU}(1,1)$, isomorphic to $\mathrm{Sl}(2,\mathbb{R})$. We do not restrict to proper $\mathrm{Sl}(2,\mathbb{R})$ representations, but want to include representations of the covering group $\widetilde{\mathrm{SU}(1,1)}$. Then the unitary representations with Casimir $c_2 = -\tau(\tau+1)$ are given by:

- the trivial representation T_0.
- the principal continuous representations $T_{(i\rho - \frac{1}{2}, \epsilon)}$, $\rho \in R$ and $\epsilon \in \{0, 1/2\}$.

- the principle discrete representations T_l^+ with lowest weight $-l > 0$ and $c_2 = -l(l+1)$.
- the principle discrete representations T_l^- with highest weight $l < 0$ and $c_2 = -l(l+1)$.
- the complementary representations $T_{(\tau,0)}$, $-1 < \tau < 0$, with lowest positive J_0^3 eigenvalue $0 \leq \alpha < 1$ for which we have the inequality $|\tau + 1/2| < |\alpha - 1/2|$.

We refer to [34] and [35] for more details.

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